

# Logarithms and exponentials

## Section 1: Introduction to logarithms

### Notes and Examples

These notes contain subsections on

- Indices and logarithms
- The laws of logarithms
- Exponential functions
- Solving exponential equations using logarithms
- An old practical application of logarithms (extension work)

### Indices and logarithms

The important thing to remember about logarithms is that, although they appear to be a new topic, they are simply about writing what you already know about indices in a different way.

A logarithm is a power.

So the statement  $2^3 = 8$   
can be written in the form  $\log_2 8 = 3$

In general:  $\log_a b = x \Leftrightarrow a^x = b$ .

To remember this, notice that  $a$  is both the base of the logarithm and the base of the index, and  $x$ , the logarithm, is the index. The value of  $\log_a b$  is the answer to the question: "What power must I raise  $a$  to in order to get  $b$ ?"

If you find it difficult to work out the meaning of a statement involving logarithms, it can be simpler to change the statement into the equivalent statement involving indices.



#### Example 1

- (i) Find  $\log_4 2$
- (ii) Find  $x$ , where  $\log_5 x = -\frac{1}{2}$

#### Solution

- (i) The statement  $\log_4 2 = x$  is equivalent to  $4^x = 2$ .

Since  $4^{\frac{1}{2}} = 2$ , then  $x$  must be  $\frac{1}{2}$ .

So  $\log_4 2 = \frac{1}{2}$ .

- (ii) The statement  $\log_5 x = -\frac{1}{2}$  is equivalent to  $5^{-\frac{1}{2}} = x$ .  
 So  $x = \frac{1}{\sqrt{5}}$ .

## The laws of logarithms

The laws of logarithms are:

$$\log x + \log y = \log xy$$

$$\log x - \log y = \log \frac{x}{y}$$

$$\log x^n = n \log x$$

These can be proved using the laws of indices:

First convert into index notation:  $\log_c x = a \Leftrightarrow c^a = x$

$$\log_c y = b \Leftrightarrow c^b = y$$

To prove the first law:

$$c^a c^b = xy \Leftrightarrow c^{a+b} = xy$$

$$\Leftrightarrow \log_c xy = a + b$$

$$\Leftrightarrow \log_c xy = \log_c x + \log_c y$$

Using the laws of indices

Similarly for the second law:

$$\frac{c^a}{c^b} = \frac{x}{y} \Leftrightarrow c^{a-b} = \frac{x}{y}$$

$$\Leftrightarrow \log_c \frac{x}{y} = a - b$$

$$\Leftrightarrow \log_c \frac{x}{y} = \log_c x - \log_c y$$

For the third law:

$$\log_c x^n = a \Leftrightarrow c^a = x^n$$

$$\Leftrightarrow c^{a/n} = x$$

$$\Leftrightarrow \log_c x = \frac{a}{n}$$

$$\Leftrightarrow n \log_c x = a$$

$$\Leftrightarrow n \log_c x = \log_c x^n$$

As the first two laws of indices require the indices to have the same base, then the first two laws of logarithms require the logarithms to have the same base.



### Example 2

- (i) Write  $\log \frac{x^3 y}{\sqrt{z}}$  in terms of  $\log x$ ,  $\log y$  and  $\log z$ .
- (ii) Write  $2\log a - \log b - \frac{1}{3}\log c$  as a single logarithm.

### Solution

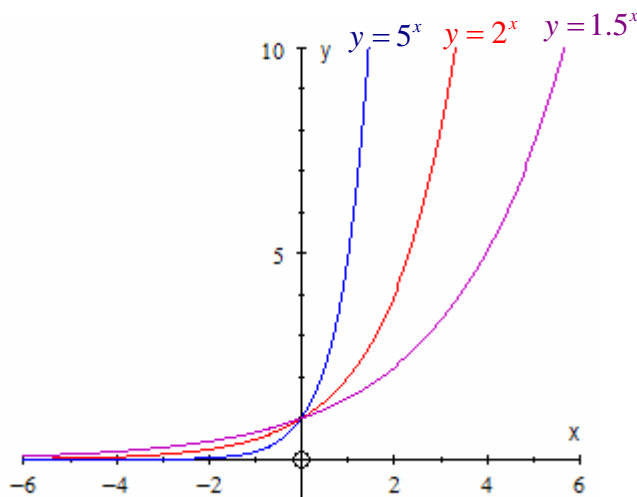
(i) 
$$\begin{aligned}\log \frac{x^3 y}{\sqrt{z}} &= \log x^3 + \log y - \log \sqrt{z} \\ &= \log x^3 + \log y - \log z^{1/2} \\ &= 3\log x + \log y - \frac{1}{2}\log z\end{aligned}$$

(ii) 
$$\begin{aligned}2\log a - \log b - \frac{1}{3}\log c &= \log a^2 - \log b - \log c^{1/3} \\ &= \log a^2 - (\log b + \log \sqrt[3]{c}) \\ &= \log \frac{a^2}{b\sqrt[3]{c}}\end{aligned}$$



## Exponential functions

An exponential function is any function of the form  $y = a^x$ . The graphs below show some different exponential functions.



Exponential functions are the inverse of logarithm functions: the function  $y = a^x$  is the inverse of the function  $y = \log_a x$ . This relationship is very useful for solving equations, as shown below.

## Solving exponential equations using logarithms

Logarithms are very useful for solving equations where the unknown variable is an index, such as the equation  $2^x = 10$ .

Many equations are solved using inverse functions, for example you solve the equation  $x + 3 = 5$ , in which addition is applied to the unknown variable, by subtracting 3 from each side. Similarly you solve the equation  $x^2 = 10$  by using the square root function, which is the inverse of the square function.

An equation like  $2^x = 10$  involves an exponential function of  $x$ . So to solve this equation, it follows that you need to use the inverse of the exponential function, which is the logarithm function. This is shown in the next example.



### Example 3

Solve the following equations.

- (i)  $2^x = 10$
- (ii)  $3^{2x-1} = 4$
- (iii)  $0.2^{1-x} = 2$



### Solution

- (i)  $2^x = 10$

$$\log 2^x = \log 10$$

$$x \log 2 = \log 10$$

$$x = \frac{\log 10}{\log 2} = 3.32$$

- (ii)  $3^{2x-1} = 4$

$$\log 3^{2x-1} = \log 4$$

$$(2x-1) \log 3 = \log 4$$

$$2x-1 = \frac{\log 4}{\log 3}$$

$$x = \frac{1}{2} \left( \frac{\log 4}{\log 3} + 1 \right) = 1.13$$

- (iii)  $0.2^{1-x} = 2$

$$\log 0.2^{1-x} = \log 2$$

$$(1-x) \log 0.2 = \log 2$$

$$1-x = \frac{\log 2}{\log 0.2}$$

$$x = 1 - \frac{\log 2}{\log 0.2} = 1.43$$

## An old practical application of logarithms



Before calculators existed, logarithms were used to make calculations easier. For example, suppose you had to divide 1432627 by 967253. You could do this by long division, but it would take a long time and the chances of making a mistake would be quite high. So you would apply the second law of logarithms:

$$\log (1432627 \div 967253) = \log 1432627 - \log 967253$$

To do the calculation, you would have to find the log to base 10 of the two numbers, subtract the results, and then find the inverse log of the answer.

You would have to find the values of  $\log 1432627$  and  $\log 967253$  from a book of tables. Unfortunately most tables would only tell you the values of  $\log x$  for values of  $x$  between 10 and 99. So you would then use the fact that

$$\begin{aligned}\log 1432627 &= \log(14.32627 \times 100000) \\ &= \log 14.32627 + \log 100000 \\ &= \log 14.32627 + 5\end{aligned}$$

You would then use the tables to find the value of  $\log 14.33$  (which is as accurate as most tables would give you). This would give a value for  $\log 1432627$  of 6.1562.

You would then go through a similar process to find  $\log 967253$ .

$$\begin{aligned}\log 967253 &= \log 96.7253 + \log 10000 \\ &= 1.9855 + 4 \\ &= 5.9855\end{aligned}$$

Next you would subtract these two logarithms (without a calculator of course!), giving 0.1707.

Now you would have to find the number whose logarithm is 0.1707. Inverse log tables usually give values between 1 and 10.

$$\begin{aligned}0.1707 &= 1.1707 - 1 \\ &= \log 14.81 - \log 10 \\ &= \log \frac{14.81}{10} \\ &= \log 1.481\end{aligned}$$

So  $1432627 \div 967253 = 1.481$ .

Most pupils did not understand the theory behind these calculations; they just followed a set of instructions to use the tables of logarithms and work out the calculation. Even after calculators became widely available, it was several years before this technique was removed from examination syllabuses! Logarithms were also the basis of slide rules, which were also used before calculators existed to work out calculations quickly.